Sistem linearnih jednačina

Sistem od m jednačina sa n nepoznatih zovemo sistem linearnih jednačina

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned}$$

Sisteme linearnih jednačina možemo riješiti:

a) Gausovom metodom
b) Kramerovom metodom (metoda determinanti)
c) Matričnom metodom
d) Kroneker-Kapelijevom metodom

1. Gausovom metodom riješiti sistem jednačina

$$\begin{aligned} x_1 + x_2 - 2x_3 + 4x_4 &= -1 \quad (1) \\ 3x_1 + 2x_2 - x_3 + 3x_4 &= 0 \quad (2) \\ 2x_1 - x_2 + 3x_3 - x_4 &= 9 \quad (3) \\ 5x_1 - 2x_2 + x_3 - 2x_4 &= 9 \quad (4) \end{aligned}$$

Rj.

$(1)+2\cdot(4)$: $11x_1 - 3x_2 = 17$
$(2)+(4)$: $8x_1 + x_4 = 9$
$(3)-3\cdot(4)$: $-13x_1 + 5x_2 + 5x_4 = -18$

$3x_2 = 11x_1 - 17 \Rightarrow x_2 = \frac{1}{3}(11x_1 - 17)$
$x_4 = -8x_1 + 9$
$-13x_1 + 5x_2 + 5x_4 = -18$

$-13x_1 + \frac{5}{3}(11x_1 - 17) + 5(-8x_1 + 9) = -18$
$-13x_1 + \frac{55}{3}x_1 - \frac{85}{3} - 40x_1 + 45 = -18$
$-53x_1 + \frac{55}{3}x_1 = -63 + \frac{85}{3} \quad / \cdot 3$
$-159x_1 + 55x_1 = -189 + 85$
$-104x_1 = -104$
$x_1 = 1$

$x_2 = \frac{1}{3}(11x_1 - 17) = \frac{1}{3}(11 - 17) = -2$
$x_4 = -8x_1 + 9 = 1$
$x_1 + x_2 - 2x_3 + 4x_4 = -1$
$-2x_3 = -1 + 2 - 4 - 1$
$-2x_3 = -4$
$x_3 = 2$

Rješenje sistema je $x_1 = 1$, $x_2 = -2$, $x_3 = 2$, $x_4 = 1$

2. Gausovom metodom riješiti sistem jednačina

$$\begin{aligned} 2x_1 + 3x_2 - 5x_3 + x_4 - x_5 &= 0 \\ x_1 + 2x_2 + 3x_3 + 2x_4 + 2x_5 &= 3 \\ 4x_1 + 7x_2 + x_3 + 5x_4 + 3x_5 &= 6 \\ 5x_1 + 9x_2 + 4x_3 + 7x_4 + 5x_5 &= 9 \end{aligned}$$

#) Riješiti sistem linearnih jednačina

$$\begin{aligned} 2x_1 - 2x_2 + 2x_3 + 3x_4 &= 1 \\ -2x_1 + x_2 - x_3 - 4x_4 &= 0 \\ 2x_1 - 3x_2 + 3x_3 + 2x_4 &= 2 \\ -x_2 + x_3 - x_4 &= 1 \end{aligned}$$

Rj. Riješimo sistem Gausovom metodom:

$$\begin{aligned} 2x_1 - 2x_2 + 2x_3 + 3x_4 &= 1 \quad (a) \\ -2x_1 + x_2 - x_3 - 4x_4 &= 0 \quad (b) \\ 2x_1 - 3x_2 + 3x_3 + 2x_4 &= 2 \quad (c) \\ -x_2 + x_3 - x_4 &= 1 \quad (d) \end{aligned}$$

(a): $2x_1 - 2x_2 + 2x_3 + 3x_4 = 1$
$(b)+(a)$: $-x_2 + x_3 - x_4 = 1$
$(c)-(a)$: $-x_2 + x_3 - x_4 = 1$
$-x_2 + x_3 - x_4 = 1$

$$\begin{aligned} 2x_1 - 2x_2 + 2x_3 + 3x_4 &= 1 \\ -x_2 + x_3 - x_4 &= 1 \end{aligned}$$

Imamo dvije linearne jednačine sa četri nepoznate $\Rightarrow$
$\Rightarrow$ dvije promjenjive uzimamo proizvoljno npr. $x_3 = s$, $x_4 = t$

$x_2 = s - t - 1$

$2x_1 = 1 + 2x_2 - 2x_3 - 3x_4$
$2x_1 = 1 + \underline{2s} - \underline{\underline{2t}} - 2 - \underline{2s} - \underline{\underline{3t}}$
$2x_1 = -5t - 1$
$x_1 = -\frac{5}{2}t - \frac{1}{2}$

Rješenje sistema linearnih jednačina je
$\left(-\frac{5}{2}t - \frac{1}{2},\ s - t - 1,\ s,\ t\right)$

Cramerovo pravilo (metoda determinanti)

Rješavamo sistem oblika $A\cdot x = b$ gdje je $A=[a_{ij}]_{n\times n}$, $x=\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}$, $b=\begin{bmatrix}b_1\\b_2\\\vdots\\b_n\end{bmatrix}$. D_k determinanta koja se dobije od D ($D=\det A$) kada se umjesto k-te kolone u D stave slobodni članovi $\begin{bmatrix}b_1\\b_2\\\vdots\\b_n\end{bmatrix}$.

a) za $D\neq 0$ sistem ima jedinstveno rješenje $x=\frac{D_x}{D}$, $y=\frac{D_y}{D}$, $z=\frac{D_z}{D}$

b) za $D=0$; ($D_x\neq 0$ ili $D_y\neq 0$ ili $D_z\neq 0$) sistem nema nijedno rješenje

c) za $D=D_x=D_y=D_z$ ne možemo ništa zaključiti (sistem može imati ∞ mnogo rješenja ili nemati nijedno rješenje) (potrebna su dalja ispitivanja)

) Metodom determinanti riješiti sistem jednačina

$$2x-y-z=4$$
$$3x+4y-2z=11$$
$$3x-2y+4z=11.$$

Rj.

$$D=\begin{vmatrix}2&-1&-1\\3&4&-2\\3&-2&4\end{vmatrix}\ \underset{III_v+I_v\cdot 4}{\overset{II_v+I_v\cdot(-2)}{=\!=}}\ \begin{vmatrix}2&-1&-1\\-1&6&0\\11&-6&0\end{vmatrix}=(-1)\begin{vmatrix}-1&6\\11&-6\end{vmatrix}=-(6-66)=60$$

$$D_x=\begin{vmatrix}4&-1&-1\\11&4&-2\\11&-2&4\end{vmatrix}\ \underset{III_v+I_v\cdot 4}{\overset{II_v-I_v\cdot 2}{=\!=}}\ \begin{vmatrix}4&-1&-1\\3&6&0\\27&-6&0\end{vmatrix}=(-1)\begin{vmatrix}3&6\\27&-6\end{vmatrix}=-(-18-162)=180$$

$$D_y=\begin{vmatrix}2&4&-1\\3&11&-2\\3&11&4\end{vmatrix}\ \underset{II_k+III_k\cdot 4}{\overset{I_k+III_k\cdot 2}{=\!=}}\ \begin{vmatrix}0&0&-1\\-1&3&-2\\11&27&4\end{vmatrix}=(-1)\begin{vmatrix}-1&3\\11&27\end{vmatrix}=-(-27-33)=60$$

$$D_z=\begin{vmatrix}2&-1&4\\3&4&11\\3&-2&11\end{vmatrix}\ \underset{III_v-I_v\cdot 2}{\overset{II_v+I_v\cdot 4}{=\!=}}\ \begin{vmatrix}2&-1&4\\11&0&27\\-1&0&3\end{vmatrix}=\begin{vmatrix}11&27\\-1&3\end{vmatrix}=3\begin{vmatrix}11&9\\-1&1\end{vmatrix}=3(11+9)=60$$

$$x=\frac{D_x}{D}=\frac{180}{60}=3;\quad y=\frac{D_y}{D}=\frac{60}{60}=1;\quad z=\frac{D_z}{D}=\frac{60}{60}=1$$

Rješenje sistema je $x=3$, $y=1$ i $z=1$.

)^v Metodom determinanti riješiti sistem jednačina:

$$2x+4y-5z=-5$$
$$-x-y+z=0$$
$$2x+y-z=1.$$

Rj. $x=1$, $y=2$, $z=3$

) Riješiti sistem jednačina i diskutovati rješenja u zavisnosti od parametra λ:

$$(\lambda-2)x-3y+2z=1$$
$$3x-3y+(\lambda-3)z=1$$
$$x-y+2z=-1.$$

Rj.

$$D=\begin{vmatrix}\lambda-2&-3&2\\3&-3&\lambda-3\\1&-1&2\end{vmatrix}\ \underset{III_k+II_k\cdot 2}{\overset{I_k+II_k}{=\!=}}\ \begin{vmatrix}\lambda-5&-3&-4\\0&-3&\lambda-9\\0&-1&0\end{vmatrix}=(\lambda-5)\begin{vmatrix}-3&\lambda-9\\-1&0\end{vmatrix}=(\lambda-5)(\lambda-9)$$

$$D_x=\begin{vmatrix}1&-3&2\\1&-3&\lambda-3\\-1&-1&2\end{vmatrix}\ \underset{II_v+III_v}{\overset{I_v+III_v}{=\!=}}\ \begin{vmatrix}0&-4&4\\0&-4&\lambda-1\\-1&-1&2\end{vmatrix}=(-1)\begin{vmatrix}-4&4\\-4&\lambda-1\end{vmatrix}=(-1)(-4)\begin{vmatrix}1&4\\1&\lambda-1\end{vmatrix}=4(\lambda-5)$$

$$D_y=\begin{vmatrix}\lambda-2&1&2\\3&1&\lambda-3\\1&-1&2\end{vmatrix}\ \underset{II_v+III_v}{\overset{I_v+III_v}{=\!=}}\ \begin{vmatrix}\lambda-1&0&4\\4&0&\lambda-1\\1&-1&2\end{vmatrix}=\begin{vmatrix}\lambda-1&4\\4&\lambda-1\end{vmatrix}=(\lambda-1)^2-4=(\lambda-1-4)(\lambda-1+4)=(\lambda-5)(\lambda+3)$$

$$D_z=\begin{vmatrix}\lambda-2&-3&1\\3&-3&1\\1&-1&-1\end{vmatrix}\ \overset{I_k+II_k}{=\!=}\ \begin{vmatrix}\lambda-5&-3&1\\0&-3&1\\0&-1&-1\end{vmatrix}=(\lambda-5)\begin{vmatrix}-3&1\\-1&-1\end{vmatrix}=4(\lambda-5)$$

Diskusija

1° $\lambda\neq 5$ i $\lambda\neq 9$ ($D\neq 0$) Sistem ima jedinstveno rješenje

$$x=\frac{D_x}{D}=\frac{4(\lambda-5)}{(\lambda-5)(\lambda-9)}=\frac{4}{\lambda-9},\quad y=\frac{D_y}{D}=\frac{\lambda+3}{\lambda-9},\quad z=\frac{D_z}{D}=\frac{4}{\lambda-9}$$

2° $\lambda=9$

$D=0$, $D_x\neq 0$ ⇒ sistem nema rješenja

3° $\lambda=5$ ⇒ $D=D_x=D_y=D_z=0$ na osnovu Cramerovog pravila ne možemo ništa zaključiti. A trebamo je uraditi sistem na drugi način.

za $\lambda=5$ sistem postaje

$$3x-3y+2z=1 \quad (1)$$
$$3x-3y+2z=1 \quad (2)$$
$$x-y+2z=-1 \quad (3)$$

(1) ≡ (2)

(2) − (3): $2x-2y=2$

$x=y+1$

$x-y+2z=-1$

$y+1-y+2z=-1$

$2z=-2$

$z=-1$

sistem ima beskonačno mnogo rješenja koji su oblika $(t+1,\ t,\ -1)$, $t\in\mathbb{R}$

)^v Riješiti sistem jednačina i diskutovati rješenja u zavisnosti od parametra λ:

$$(\lambda+4)x+y+z=2$$
$$x+y+z=\lambda+5$$
$$3x+3y+(\lambda+7)z=3$$

Rj. $D=(\lambda+4)(\lambda+3)$

$D_x=-(\lambda+4)(\lambda+3)$

$D_y=(\lambda+3)(\lambda+4)(\lambda+8)$

$D_z=-3(\lambda+3)(\lambda+4)$

$(t, 5-t, -3)$, $t\in\mathbb{R}$

$(-1, 2-s, s)$, $s\in\mathbb{R}$

Riješiti sistem jednačina i diskutovati rješenja sistema u zavisnosti od parametra λ

$$x+y+z=4$$
$$x+\lambda y+z=3$$
$$x+2\lambda y+z=4.$$

Rj. Sistem rješavamo Cramerovom metodom

$$D=\begin{vmatrix}1&1&1\\1&\lambda&1\\1&2\lambda&1\end{vmatrix}\overset{II_v-III_v}{=\!=}\begin{vmatrix}1&1&1\\0&-\lambda&0\\1&2\lambda&1\end{vmatrix}=-\lambda\begin{vmatrix}1&1\\1&1\end{vmatrix}=0$$

$$D_x=\begin{vmatrix}4&1&1\\3&\lambda&1\\4&2\lambda&1\end{vmatrix}\underset{III_v-II_v}{\overset{I_v-II_v}{=\!=}}\begin{vmatrix}1&1-\lambda&0\\3&\lambda&1\\1&\lambda&0\end{vmatrix}=-\begin{vmatrix}1&1-\lambda\\1&\lambda\end{vmatrix}=-(\lambda-(1-\lambda))=1-\lambda-\lambda=1-2\lambda$$

$$D_y=\begin{vmatrix}1&4&1\\1&3&1\\1&4&1\end{vmatrix}\overset{III_k-I_k}{=\!=}\begin{vmatrix}1&4&0\\1&3&0\\1&4&0\end{vmatrix}=0$$

$$D_z=\begin{vmatrix}1&1&4\\1&\lambda&3\\1&2\lambda&4\end{vmatrix}\underset{III_v-II_v}{\overset{I_v-II_v}{=\!=}}\begin{vmatrix}0&1-\lambda&1\\1&\lambda&3\\0&\lambda&1\end{vmatrix}=-\begin{vmatrix}1-\lambda&1\\\lambda&1\end{vmatrix}=-(1-\lambda-\lambda)=2\lambda-1$$

Kako je $D=0$ to sistem može da ima beskonačno mnogo rješenja ili da nema rješenja.

1° $\lambda=\frac{1}{2}$

$D=0,\ D_x=0,\ D_y=0,\ D_z=0$

Sistem ćemo riješiti Gauss-ovom metodom

$$x+y+z=4$$
$$x+\tfrac{1}{2}y+z=3 \quad /\cdot 2$$
$$x+y+z=4$$

$$x+y+z=4 \quad (1)$$
$$2x+y+2z=6 \quad (2)$$
$$(2)-(1):\ x+z=2$$
$$x=2-z$$

$$x+y+z=4$$
$$2-z+y+z=4$$
$$y=2$$

Za $\lambda=\frac{1}{2}$ sistem ima ∞ mnogo rješenja koja su oblika $(2-t, 2, t)$ gdje je $t\in\mathbb{R}$.

2° $\lambda\neq\frac{1}{2}$

$D=0,\ D_x\neq 0 \implies$ sistem za $\lambda\neq\frac{1}{2}$ nema rješenja

Odrediti vrijednost parametra k tako da sistem

$$8z-3x-6y=kx$$
$$2x+y+4z=ky$$
$$4x+3y+z=kz$$

ima beskonačno mnogo rješenja. Zatim naći ta rješenja za najveću dobijenu vrijednost parametra k.

Rj. Nepoznate sa desne strane prebacimo na lijevu i grupirajmo vrijednosti uz x, y i z.

$$(-3-k)x-6y+8z=0$$
$$2x+(1-k)y+4z=0$$
$$4x+3y+(1-k)z=0$$

Ovo je homogeni sistem linearnih jednačina. Trivijalno rješenje je $(0,0,0)$. Sistem ima beskonačno mnogo rješenja ako je $D=0$.

$$\begin{vmatrix}-3-k&-6&8\\2&1-k&4\\4&3&1-k\end{vmatrix}=0$$

$$I_k+III_k:\ \begin{vmatrix}5-k&-6&8\\6&1-k&4\\5-k&3&1-k\end{vmatrix}=0$$

$$I_v-III_v:\ \begin{vmatrix}0&-9&7+k\\6&1-k&4\\5-k&3&1-k\end{vmatrix}=0$$

$$6[\overbrace{(-9)(1-k)}^{-9+9k}-\overbrace{3(7+k)}^{-21-3k}]+(5-k)[(-9)\cdot 4-\overbrace{(7+k)(1-k)}^{\substack{7-6k-k^2\\7-7k+k-k^2}}]=0$$

$$(-6)(6k-30)+(5-k)(\overbrace{-36-7}^{-43}+6k+k^2)=0$$

$$-36k+180+(-215)+30k+5k^2+43k-6k^2-k^3=0$$

$$-k^3-k^2+37k-35=0 \quad /\cdot(-1)$$

$$k^3+k^2-37k+35=0$$
$$k^3-k^2+2k^2-2k-35k+35=0$$
$$k^2(1-k)+2k(k-1)-35(k-1)=0$$

$$(k-1)(k^2+2k-35)=0$$
$$(k-1)(k+7)(k-5)=0$$
$$k_1=1,\ k_2=-7,\ k_3=5$$

Za $k=5$ imamo:

$$8x+6y-8z=0 \quad \ldots(1)$$
$$2x-4y+4z=0 \quad \ldots(2)$$
$$4x+3y-4z=0 \quad \ldots(3)$$

$(1)\equiv(3)$ jer se (3) dobija dijeljenjem (1) sa 2.

$$(2)+(3):\ 6x-y=0 \implies y=6x$$

$$(2)\implies 2x-24x+4z=0$$
$$4z=22x$$
$$z=\frac{11x}{2}$$

Za $k=5$ sistem ima rješenje $(t, 6t, \frac{11t}{2})$ gdje je $t\in\mathbb{R}$ proizvoljno.

Riješiti sistem jednačina i diskutovati rješenja sistema u zavisnosti od parametra λ:

$$\begin{aligned} x - y - \lambda z &= 1 \\ (\lambda+1)y + (\lambda-1)z &= 0 \\ (\lambda+1)x - (\lambda+1)z &= 1. \end{aligned}$$

Rj.

$$D = \begin{vmatrix} 1 & -1 & -\lambda \\ 0 & \lambda+1 & \lambda-1 \\ \lambda+1 & 0 & -(\lambda+1) \end{vmatrix} \overset{III_k+I_k}{=\!=} \begin{vmatrix} 1 & -1 & 1-\lambda \\ 0 & \lambda+1 & \lambda-1 \\ \lambda+1 & 0 & 0 \end{vmatrix} = (\lambda+1)\begin{vmatrix} -1 & -(\lambda-1) \\ \lambda+1 & \lambda-1 \end{vmatrix} =$$

$$= (\lambda+1)(\lambda-1)\begin{vmatrix} -1 & -1 \\ \lambda+1 & 1 \end{vmatrix} = \lambda(\lambda-1)(\lambda+1)$$

$$D_x = \begin{vmatrix} 1 & -1 & -\lambda \\ 0 & \lambda+1 & \lambda-1 \\ 1 & 0 & -(\lambda+1) \end{vmatrix} \overset{III_v-I_v}{=\!=} \begin{vmatrix} 1 & -1 & -\lambda \\ 0 & \lambda+1 & \lambda-1 \\ 0 & 1 & -1 \end{vmatrix} = \begin{vmatrix} \lambda+1 & \lambda-1 \\ 1 & -1 \end{vmatrix} \overset{-1+\lambda+1}{=} \lambda-1-\lambda+1 = -2\lambda$$

$$D_y = \begin{vmatrix} 1 & 1 & -\lambda \\ 0 & 0 & \lambda-1 \\ \lambda+1 & 1 & -(\lambda+1) \end{vmatrix} = -(\lambda-1)\begin{vmatrix} 1 & 1 \\ \lambda+1 & 1 \end{vmatrix} = -(\lambda-1)(1-\lambda-1) = \lambda(\lambda-1)$$

$$D_z = \begin{vmatrix} 1 & -1 & 1 \\ 0 & \lambda+1 & 0 \\ \lambda+1 & 0 & 1 \end{vmatrix} = (\lambda+1)\begin{vmatrix} 1 & 1 \\ \lambda+1 & 1 \end{vmatrix} = -\lambda(\lambda+1)$$

$D=0$ akko $\lambda=0$ ili $\lambda=1$ ili $\lambda=-1$

Diskusija

1° $\lambda\neq 0$; $\lambda\neq 1$; $\lambda\neq -1$ sistem ima jedinstveno rješenje

$$x = \frac{D_x}{D} = \frac{-2\lambda}{\lambda(\lambda-1)(\lambda+1)} = \frac{-2}{(\lambda-1)(\lambda+1)}, \quad y = \frac{D_y}{D} = \frac{1}{\lambda+1}, \quad z = \frac{D_z}{D} = \frac{-1}{\lambda+1}$$

2° $\lambda=1$, $D=0$, $D_x\neq 0 \Rightarrow$ sistem nema rješenja.

3° $\lambda=-1$, $D=0$, $D_x\neq 0 \Rightarrow$ sistem nema rješenja

4° $\lambda=0$, $D=D_x=D_y=D_z=0$ iz ovoga ne možemo ništa zaključiti

Za $\lambda=0$ sistem postaje

$$\begin{aligned} x - y \quad &= 1 \quad (1) \\ y - z &= 0 \quad (2) \\ x \quad - z &= 1 \quad (3) \end{aligned}$$

(1): $x-y=1$

$(2)-(3)$: $-x+y=-1$

$x=y+1$

$x-z=1$

$-z=-(y+1)+1$

$-z=-y$

$z=y$

Sistem ima ∞ mnogo rješenja $(t+1, t, t)$, $t\in R$

Riješiti sistem jednačina i diskutovati rješenja sistema u zavisnosti od parametra a:

$$\begin{aligned} x+y-z &= 0 \\ x-y+az &= 1 \\ -x-3y+(a+2)z &= a^2. \end{aligned}$$

Rj.

$$D = \begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & a \\ -1 & -3 & a+2 \end{vmatrix} \underset{II_k+III_k}{\overset{I_k+III_k}{=\!=}} \begin{vmatrix} 0 & 0 & -1 \\ a+1 & a-1 & a \\ a+1 & a-1 & a+2 \end{vmatrix} = (-1)\begin{vmatrix} a+1 & a-1 \\ a+1 & a-1 \end{vmatrix} = 0$$

$$D_x = \begin{vmatrix} 0 & 1 & -1 \\ 1 & -1 & a \\ a^2 & -3 & a+2 \end{vmatrix} \overset{II_k+III_k}{=\!=} \begin{vmatrix} 0 & 0 & -1 \\ 1 & a-1 & a \\ a^2 & a-1 & a+2 \end{vmatrix} = (-1)\begin{vmatrix} 1 & a-1 \\ a^2 & a-1 \end{vmatrix} = (-1)(a-1)\begin{vmatrix} 1 & 1 \\ a^2 & 1 \end{vmatrix}$$

$$= (-1)(a-1)(1-a^2) = (a-1)(a^2-1) = (a-1)^2(a+1)$$

$$D_y = \begin{vmatrix} 1 & 0 & -1 \\ 1 & 1 & a \\ -1 & a^2 & a+2 \end{vmatrix} \overset{I_k+III_k}{=\!=} \begin{vmatrix} 0 & 0 & -1 \\ a+1 & 1 & a \\ a+1 & a^2 & a+2 \end{vmatrix} = (-1)\begin{vmatrix} a+1 & 1 \\ a+1 & a^2 \end{vmatrix} = (-1)(a+1)\begin{vmatrix} 1 & 1 \\ 1 & a^2 \end{vmatrix} = (-1)(a+1)(a^2-1)$$

$$= (-1)(a-1)(a+1)^2$$

$$D_z = \begin{vmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ -1 & -3 & a^2 \end{vmatrix} \overset{I_k-II_k}{=\!=} \begin{vmatrix} 0 & 1 & 0 \\ 2 & -1 & 1 \\ 2 & -3 & a^2 \end{vmatrix} = (-1)\begin{vmatrix} 2 & 1 \\ 2 & a^2 \end{vmatrix} = (-1)(2a^2-2) = (-2)(a+1)(a-1)$$

Diskusija

$D=0$ $\forall a\in R$

1° $a\neq 1$; $a\neq -1$

$D=0$; $D_x\neq 0$ sistem nema rješenja

2° $a=1$

$D=D_x=D_y=D_z=0$, sistem postaje

$$\begin{aligned} x+y-z &= 0 \quad (1) \\ x-y+z &= 1 \quad (2) \\ -x-3y+3z &= 1 \quad (3) \end{aligned}$$

$(1)+(3)$: $-2y+2z=1$

$(2)+(3)$: $-4y+4z=2$

$2z=2y+1$

$z=y+\frac{1}{2}$

$x=z-y$

$x=\frac{1}{2}$

Sistem ima ∞ mnogo rješenja oblika $(\frac{1}{2}, t, t+\frac{1}{2})$ gdje je $t\in R$.

3° $a=-1$

$D=D_x=D_y=D_z=0$, sistem postaje

$$\begin{aligned} x+y-z &= 0 \quad I \\ x-y-z &= 1 \quad II \\ -x-3y+z &= 1 \quad III \end{aligned}$$

$(I)+(III)$: $-2y=1$

$(II)+(III)$: $-4y=2$

$y=-\frac{1}{2}$

$(I)+(II)$: $2x-2z=1$

$(III)-3\cdot(II)$: $-4x+4z=2$

$2x=2z+1$

$x=z+\frac{1}{2}$

Sistem ima ∞ mnogo rješenja oblika $(t+\frac{1}{2}, -\frac{1}{2}, t)$, $t\in Z$

(#) Diskutovati rješenja sistema u zavisnosti od parametra λ:

$$2x - \lambda y + 2z = 1$$
$$x + y + 2z = 0$$
$$-x + (-\lambda-3)y - 4z = \lambda .$$

Rj. Sistem ćemo riješiti Cramerovim pravilima.

$$D = \begin{vmatrix} 2 & -\lambda & 2 \\ 1 & 1 & 2 \\ -1 & -\lambda-3 & -4 \end{vmatrix} \overset{I_k - II_k}{\underset{III_k - II_k\cdot 2}{=}} \begin{vmatrix} 2+\lambda & -\lambda & 2\lambda+2 \\ 0 & 1 & 0 \\ \lambda+2 & -\lambda-3 & 2\lambda+2 \end{vmatrix} = \begin{vmatrix} \lambda+2 & 2\lambda+2 \\ \lambda+2 & 2\lambda+2 \end{vmatrix} = (\lambda+2)\begin{vmatrix} 1 & 2\lambda+2 \\ 1 & 2\lambda+2 \end{vmatrix} = 0$$

$$D_x = \begin{vmatrix} 1 & -\lambda & 2 \\ 0 & 1 & 2 \\ \lambda & -\lambda-3 & -4 \end{vmatrix} \overset{III_k - II_k\cdot 2}{=} \begin{vmatrix} 1 & -\lambda & 2\lambda+2 \\ 0 & 1 & 0 \\ \lambda & -\lambda-3 & 2\lambda+2 \end{vmatrix} = \begin{vmatrix} 1 & 2\lambda+2 \\ \lambda & 2\lambda+2 \end{vmatrix} = (2\lambda+2)\begin{vmatrix} 1 & 1 \\ \lambda & 1 \end{vmatrix}$$
$$= (2\lambda+2)(1-\lambda) = 2(\lambda+1)(1-\lambda)$$

$$D_y = \begin{vmatrix} 2 & 1 & 2 \\ 1 & 0 & 2 \\ -1 & \lambda & -4 \end{vmatrix} \overset{III_k - I_k\cdot 2}{=} \begin{vmatrix} 2 & 1 & -2 \\ 1 & 0 & 0 \\ -1 & \lambda & -2 \end{vmatrix} = (-1)\begin{vmatrix} 1 & -2 \\ \lambda & -2 \end{vmatrix} = (-1)(-2)\begin{vmatrix} 1 & 1 \\ \lambda & 1 \end{vmatrix} = 2(1-\lambda)$$

$$D_z = \begin{vmatrix} 2 & -\lambda & 1 \\ 1 & 1 & 0 \\ -1 & -\lambda-3 & \lambda \end{vmatrix} \overset{I_k - II_k}{=} \begin{vmatrix} 2+\lambda & -\lambda & 1 \\ 0 & 1 & 0 \\ \lambda+2 & -\lambda-3 & \lambda \end{vmatrix} = \begin{vmatrix} \lambda+2 & 1 \\ \lambda+2 & \lambda \end{vmatrix} = (\lambda+2)\begin{vmatrix} 1 & 1 \\ 1 & \lambda \end{vmatrix} = (\lambda+2)(\lambda-1)$$

Diskusija:

$D=0$, $D_x = 2(1+\lambda)(1-\lambda)$, $D_y = 2(1-\lambda)$, $D_z = (\lambda+2)(\lambda-1)$

1° $\lambda \neq -1$; $\lambda \neq 1$; $\lambda \neq -2$
imamo $D=0$; $D_x \neq 0$ sistem nema rješenja

2° $\lambda = -2$ imamo $D=0$; $D_x \neq 0$ sistem nema rješenja

3° $\lambda = -1$ imamo $D=0$, $D_x = 0$, $D_y \neq 0$ sistem nema rješenja

4° $\lambda = 1$ imamo $D = D_x = D_y = D_z = 0$ sistem je potrebno ispitati na drugi način.

Za $\lambda=1$ sistem postaje

$$2x - y + 2z = 1 \quad /\cdot 4$$
$$x + y + 2z = 0 \quad /\cdot 4$$
$$-x - 4y - 4z = 1$$

$$8x - 4y + 8z = 4 \quad (1)$$
$$4x + 4y + 8z = 0 \quad (2)$$
$$-x - 4y - 4z = 1 \quad (3)$$

(1)+(2): $12x + 16z = 4$

(3)+(2): $3x + 4z = 1$

$3x = 1 - 4z$

$x = \frac{1-4z}{3}$

$y = -x - 2z$

$y = \frac{4z-1}{3} - \frac{6z}{3}$

Sistem ima ∞ mnogo rješenja, oblika $\left(\frac{1-4t}{3}, \frac{-2t-1}{3}, t\right)$ $t \in \mathbb{R}$

(#) Riješiti sistem jednačina i diskutovati rješenja u zavisnosti od parametra

$$x + y + bz = 1 - b$$
$$x - by - z = 2$$
$$bx - y + z = 2b .$$

Rj. Rješavamo sistem Cramerovom metodom

$$D = \begin{vmatrix} 1 & 1 & b \\ 1 & -b & -1 \\ b & -1 & 1 \end{vmatrix} \overset{I_k + III_k}{=} \begin{vmatrix} b+1 & 1 & b \\ 0 & -b & -1 \\ b+1 & -1 & 1 \end{vmatrix} = (b+1)\begin{vmatrix} 1 & 1 & b \\ 0 & -b & -1 \\ 1 & -1 & 1 \end{vmatrix} \overset{I_v - III_v}{=}$$
$$= (b+1)\begin{vmatrix} 0 & 2 & b-1 \\ 0 & -b & -1 \\ 1 & -1 & 1 \end{vmatrix} = (b+1)\begin{vmatrix} 2 & b-1 \\ -b & -1 \end{vmatrix} = (b+1)\Big[\overbrace{-2+(b^2-b)}^{b^2-b-2}\Big] =$$
$$= (b+1)(b+1)(b-2)$$

$$D_x = \begin{vmatrix} 1-b & 1 & b \\ 2 & -b & -1 \\ 2b & -1 & 1 \end{vmatrix} \overset{I_v + III_v}{=} \begin{vmatrix} b+1 & 0 & b+1 \\ 2 & -b & -1 \\ 2b & -1 & 1 \end{vmatrix} = (b+1)\begin{vmatrix} 1 & 0 & 1 \\ 2 & -b & -1 \\ 2b & -1 & 1 \end{vmatrix} =$$
$$\overset{I_k - III_k}{=} (b+1)\begin{vmatrix} 0 & 0 & 1 \\ 3 & -b & -1 \\ 2b-1 & -1 & 1 \end{vmatrix} = (b+1)\begin{vmatrix} 3 & -b \\ 2b-1 & -1 \end{vmatrix} = (b+1)(\overbrace{-3+2b^2-b}^{2b^2-b-3}) =$$

$D = 1 + 24 = 25$, $b_{1,2} = \frac{1\pm 5}{4}$

$$= (b+1)\cdot 2\left(b - \frac{3}{2}\right)(b+1)$$
$$= (b+1)(2b-3)(b+1)$$

$$D_y = \begin{vmatrix} 1 & 1-b & b \\ 1 & 2 & -1 \\ b & 2b & 1 \end{vmatrix} \overset{I_k + III_k}{=} \begin{vmatrix} b+1 & 1-b & b \\ 0 & 2 & -1 \\ b+1 & 2b & 1 \end{vmatrix} = (b+1)\begin{vmatrix} 1 & 1-b & b \\ 0 & 2 & -1 \\ 1 & 2b & 1 \end{vmatrix} \overset{III_v - I_v}{=}$$
$$= (b+1)\begin{vmatrix} 1 & 1-b & b \\ 0 & 2 & -1 \\ 0 & 3b-1 & 1-b \end{vmatrix} = (b+1)\begin{vmatrix} 2 & -1 \\ 3b-1 & 1-b \end{vmatrix} = (b+1)(2-2b+3b-1) =$$
$$= (b+1)(b+1)$$

$$D_z = \begin{vmatrix} 1 & 1 & 1-b \\ 1 & -b & 2 \\ b & -1 & 2b \end{vmatrix} \overset{I_v + III_v}{=} \begin{vmatrix} b+1 & 0 & b+1 \\ 1 & -b & 2 \\ b & -1 & 2b \end{vmatrix} = (b+1)\begin{vmatrix} 1 & 0 & 1 \\ 1 & -b & 2 \\ b & -1 & 2b \end{vmatrix} \overset{I_k - III_k}{=}$$
$$= (b+1)\begin{vmatrix} 0 & 0 & 1 \\ -1 & -b & 2 \\ -b & -1 & 2b \end{vmatrix} = (b+1)\begin{vmatrix} -1 & -b \\ -b & -1 \end{vmatrix} = (b+1)(1-b^2) = -(b+1)(b^2-1)$$
$$= -(b+1)(b-1)(b+1)$$

Diskusija: a) $D \neq 0$ tj. $b \neq -1$; $b \neq 2$

sistem ima jedinstveno rješenje $x = \frac{D_x}{D} = \frac{(2b-3)(b+1)^2}{(b+1)^2(b-2)} = \frac{2b-3}{b-2}$

$y = \frac{D_y}{D} = \frac{(b+1)^2}{(b+1)^2(b-2)} = \frac{1}{b-2}$; $z = \frac{D_z}{D} = \frac{-(b-1)(b+1)^2}{(b-2)(b+1)^2} = -\frac{b-1}{b-2}$

b) $b=-1 \Rightarrow D=D_x=D_y=D_z=0$ sistem trebamo rješiti na drugi način

Za $b=-1$ sistem postaje

$$\begin{aligned} x+y-z&=2\\ x+y-z&=2\\ -x-y+z&=-2 \quad /\cdot(-1) \end{aligned}$$

Sve tri jednačine su iste $\Rightarrow$ Sistem ima ∞ mnogo rješenja. Ako uzmemo $x=t$, $y=s$ Rješenja sistema su $(t, s, t+s-2)$

← dvije promjenjive uzimamo proizvoljno

c) $b=2 \Rightarrow D=0,\ D_x=9\neq 0 \Rightarrow$

Sistem za $b=2$ nema rješenja

Kroneker-Kapelijeva metoda

Neka je dat sistem linearnih jednačina $Ax=b$, gdje su

$$A=\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n}\\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn}\end{bmatrix}_{m\times n},\quad x=\begin{bmatrix} x_1\\ x_2\\ \vdots\\ x_n\end{bmatrix} \ ; \ b=\begin{bmatrix} b_1\\ b_2\\ \vdots\\ b_m\end{bmatrix}.$$

Matricu $\bar{A}=[A\,|\,b]$ zovemo proširena matrica.

Teorema (Kroneker-Kapeli):
Sistem ima jedinstveno rješenje ako i samo ako je $\text{rang}\,A=\text{rang}\,\bar{A}=n$ (n broj nepoznatih).
Ako je $\text{rang}\,A=\text{rang}\,\bar{A}<n$ tada sistem ima ∞ mnogo rješenja. ($n-\text{rang}\,A$ nepoznatih uzima se proizvoljno)
Ako je $\text{rang}\,A<\text{rang}\,\bar{A}$ tada sistem nema rješenja.

1. Kroneker-Kapelijevom metodom rješiti sistem jednačina

$$\begin{aligned} 2x+4y-5z&=-5\\ -x-y+z&=0\\ 2x+y-z&=1. \end{aligned}$$

Rj.

$$\bar{A}=[A\,|\,b]=\left[\begin{array}{ccc|c} 2 & 4 & -5 & -5\\ -1 & -1 & 1 & 0\\ 2 & 1 & -1 & 1\end{array}\right] \underset{I_v\leftrightarrow II_v}{\sim} \left[\begin{array}{ccc|c} -1 & -1 & 1 & 0\\ 2 & 4 & -5 & -5\\ 2 & 1 & -1 & 1\end{array}\right] \underset{\substack{II_v+I_v\cdot 2\\ III_v+I_v\cdot 2}}{\sim} \left[\begin{array}{ccc|c} -1 & -1 & 1 & 0\\ 0 & 2 & -3 & -5\\ 0 & -1 & 1 & 1\end{array}\right]$$

$$\underset{II_v\leftrightarrow III_v}{\sim} \left[\begin{array}{ccc|c} -1 & -1 & 1 & 0\\ 0 & -1 & 1 & 1\\ 0 & 2 & -3 & -5\end{array}\right] \underset{III_v+II_v\cdot 2}{\sim} \left[\begin{array}{ccc|c} -1 & -1 & 1 & 0\\ 0 & -1 & 1 & 1\\ 0 & 0 & -1 & -3\end{array}\right]$$

$\text{rang}\,A=\text{rang}\,\bar{A}=3$
sistem ima jedinstveno rješenje

$$\begin{aligned} -x-y+z&=0\\ -y+z&=1\\ -z&=-3\\ \hline z&=3 \end{aligned}$$

$$\begin{aligned} -x-y&=-3\\ -y&=-2\\ \hline y&=2 \end{aligned}$$

$$\begin{aligned} -x-2&=-3\\ x&=1 \end{aligned}$$

Rješenje sistema je uređena trojka $(1,2,3)$.

2. Kroneker-Kapelijevom metodom rješiti sistem jednačina

$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ 3x_1 + x_2 - x_3 &= 3 \\ 2x_1 + x_2 \phantom{{}-x_3} &= 2. \end{aligned}$$

Rj. $\bar{A} = [A|b] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 3 & 1 & -1 & 3 \\ 2 & 1 & 0 & 2 \end{array}\right] \underset{III_v - I_v \cdot 2}{\overset{II_v - I_v \cdot 3}{\sim}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -2 & -4 & 0 \\ 0 & -1 & -2 & 0 \end{array}\right] \overset{II_v \leftrightarrow III_v}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & 0 \\ 0 & -2 & -4 & 0 \end{array}\right]$

$\overset{III_v - II_v \cdot 2}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\text{rang}\, A = \text{rang}\, \bar{A} = 2 < 3$

sistem ima ∞ mnogo rješenja

3-2 nepoznatih uzimamo proizvoljno

$x_3 = t$

$-x_2 - 2x_3 = 0$

$x_1 + x_2 + x_3 = 1$

$-x_2 - 2t = 0$

$x_2 = -2t$

$x_1 - 2t + t = 1$

$x_1 = t + 1$

Sistem ima beskonačno mnogo rješenja oblika $(t+1, -2t, t)$ gdje je $t \in R$.

3. Kroneker-Kapelijevom metodom rješiti sistem jednačina

$$\begin{aligned} x + 2y + 3z &= 1 \\ 2x + 4y + 6z &= 2 \\ 3x + 6y + 9z &= 5. \end{aligned}$$

Rj. $\bar{A} = [A|b] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 4 & 6 & 2 \\ 3 & 6 & 9 & 5 \end{array}\right] \underset{III_v - I_v \cdot 3}{\overset{II_v - I_v \cdot 2}{\sim}} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{array}\right]$

$\text{rang}\, A = 1$, $\text{rang}\, \bar{A} = 2$, $\text{rang}\, A < \text{rang}\, \bar{A}$

sistem nema rješenja.

4. Kroneker-Kapelijevom metodom diskutovati rješenja sistema za razne vrijednosti parametra λ

$$\begin{aligned} \lambda x + y + z &= 1 \\ x + \lambda y + z &= 2 \\ x + y + \lambda z &= -3. \end{aligned}$$

Rj. za $\lambda \in (-\infty, -2) \cup (-2, 1) \cup (1, +\infty)$ sistem ima jedinstveno rješenje $\left(\frac{1}{\lambda-1}, \frac{2}{\lambda-1}, \frac{-3}{\lambda-1}\right)$

za $\lambda = -2$ sistem ima ∞ mnogo rješenja $\left(\frac{3t-4}{3}, \frac{3t-5}{3}, t\right)$, $t \in R$

za $\lambda = 1$ sistem nema rješenja

Riješiti sistem jednačina za razne vrijednosti parametra $\lambda \in R$:

$$\begin{aligned} 2x_1 - x_2 + 3x_3 - 7x_4 &= 15 \\ 6x_1 - 3x_2 + x_3 - 4x_4 &= 7 \\ 4x_1 - 2x_2 + 14x_3 - 31x_4 &= \lambda \end{aligned}$$

Rj. Rješimo sistem Kroneker-Kapelijevom metodom:

$\bar{C} = [C|b] = \left[\begin{array}{cccc|c} 2 & -1 & 3 & -7 & 15 \\ 6 & -3 & 1 & -4 & 7 \\ 4 & -2 & 14 & -31 & \lambda \end{array}\right] \underset{III_v - I_v \cdot 2}{\overset{II_v - I_v \cdot 3}{\sim}} \left[\begin{array}{cccc|c} 2 & -1 & 3 & -7 & 15 \\ 0 & 0 & -8 & 17 & -38 \\ 0 & 0 & 8 & -17 & \lambda-30 \end{array}\right]$

$\overset{III_v + II_v}{\sim} \left[\begin{array}{cccc|c} 2 & -1 & 3 & -7 & 15 \\ 0 & 0 & -8 & 17 & -38 \\ 0 & 0 & 0 & 0 & \lambda-68 \end{array}\right]$

1° $\lambda - 68 \neq 0$, $\lambda \neq 68$: $\text{rang}\, C = 2$, $\text{rang}\, \bar{C} = 3$, $\text{rang}\, C < \text{rang}\, \bar{C}$. Prema Kroneker-Kapelijevoj teoremi sistem nema rješenje

2° $\lambda - 68 = 0$, $\lambda = 68$: $\text{rang}\, C = \text{rang}\, \bar{C} = 2 < 4$ (broj nepoznatih)

Prema Kroneker-Kapelijevoj teoremi dvije promjenjive uzimamo proizvoljno, npr. $x_4 = t$, $x_1 = s$

$2x_1 - x_2 + 3x_3 - 7x_4 = 15$

$-8x_3 + 17x_4 = -38$

$x_4 = t$

$-8x_3 + 17t = -38$

$-8x_3 = -17t - 38$

$x_3 = \frac{17}{8}t + \frac{38}{8} = \frac{17}{8}t + \frac{19}{4}$

$x_1 = s$

$2s - x_2 + 3\left(\frac{17}{8}t + \frac{38}{8}\right) - 7t = 15$

$x_2 = \frac{51t}{8} + \frac{114}{8} + 2s - 7t - 15$

$x_2 = -\frac{5}{8}t - \frac{6}{8} + 2s$

$x_2 = 2s - \frac{5}{8}t - \frac{3}{4}$

Za $\lambda = 68$ rješenje sistema je

$\left(s, 2s - \frac{5}{8}t - \frac{3}{4}, \frac{17}{8}t + \frac{19}{4}, t\right)$, $t, s \in R$

Riješiti sistem jednačina za razne vrijednosti parametra $\lambda \in \mathbb{R}$:

$$\begin{aligned} 8x_1+12x_2+7x_3+\lambda x_4 &= 9 \\ 6x_1+9x_2+5x_3+6x_4 &= 7 \\ 4x_1+6x_2+3x_3+4x_4 &= 5 \\ 2x_1+3x_2+2x_3+2x_4 &= 2 . \end{aligned}$$

Rj. Sistem ćemo riješiti Kroneker-Kapelijevom metodom:

$$\overline{B}=[B \mid b] = \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & x_4 & \end{matrix} \\ \left[\begin{array}{cccc|c} 8 & 12 & 7 & \lambda & 9 \\ 6 & 9 & 5 & 6 & 7 \\ 4 & 6 & 3 & 4 & 5 \\ 2 & 3 & 2 & 2 & 2 \end{array}\right] \end{array} \overset{I_v \leftrightarrow IV_v}{\sim} \left[\begin{array}{cccc|c} 2 & 3 & 2 & 2 & 2 \\ 6 & 9 & 5 & 6 & 7 \\ 4 & 6 & 3 & 4 & 5 \\ 8 & 12 & 7 & \lambda & 9 \end{array}\right] \underset{IV_v - I_v \cdot 4}{\overset{II_v - I_v \cdot 3,\ III_v - I_v \cdot 2}{\sim}}$$

$$\sim \left[\begin{array}{cccc|c} 2 & 3 & 2 & 2 & 2 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & \lambda-8 & 1 \end{array}\right] \underset{IV_v - II_v}{\overset{III_v - II_v}{\sim}} \left[\begin{array}{cccc|c} 2 & 3 & 2 & 2 & 2 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda-8 & 0 \end{array}\right]$$

1° za $\lambda=8$ imamo rang B = rang $\overline{B} = 2 < 4$ pa prema Kroneker-Kapelijevoj teoremi sistem ima ∞ mnogo rješenja. Dvije promjenjive uzimamo proizvoljno npr. $x_1=t$, $x_4=s$

$$\begin{aligned} 2x_1+3x_2+2x_3+2x_4 &= 2 \\ -x_3+0x_4 &= 1 \end{aligned}$$

$x_3=-1$

$2t+3x_2-2+2s=2$

$3x_2=4-2t-2s$

$x_2=\frac{2}{3}(2-t-s)$

Rješenje sistema (za $\lambda=8$) je $(t, \frac{2}{3}(2-t-s), -1, s)$ gdje su $s,t \in \mathbb{R}$.

2° za $\lambda \neq 8$ imamo rang B = rang $\overline{B} = 3<4$ pa prema Kroneker-Kapelijevoj teoremi sistem ima ∞ mnogo rješenja. Jednu promjenjivu uzimamo proizvoljno npr. $x_2=t$.

$$\begin{aligned} 2x_1+3x_2+2x_3+2x_4 &= 2 \\ -x_3 &= 1 \\ (\lambda-8)x_4 &= 0 \end{aligned}$$

$x_4=0$

$x_3=-1$

$2x_1+3t-2=2$

$2x_1=4-3t$

$x_1=2-\frac{3}{2}t$

Rješenje sistema (za $\lambda \neq 8$) je $(2-\frac{3}{2}t, t, -1, 0)$ gdje su $t \in \mathbb{R}$.

Riješiti sistem jednačina za razne vrijednosti parametra $\lambda \in \mathbb{R}$:

$$\begin{aligned} \lambda x_1-4x_2+9x_3+10x_4 &= 11 \\ 2x_1-x_2+3x_3+4x_4 &= 5 \\ 4x_1-2x_2+5x_3+6x_4 &= 7 \\ 6x_1-3x_2+7x_3+8x_4 &= 9 \end{aligned}$$

Rj. Sistem ćemo riješiti Kroneker-Kapelijevom metodom:

$$\overline{A}=[A \mid b]=\left[\begin{array}{cccc|c} \lambda & -4 & 9 & 10 & 11 \\ 2 & -1 & 3 & 4 & 5 \\ 4 & -2 & 5 & 6 & 7 \\ 6 & -3 & 7 & 8 & 9 \end{array}\right] \overset{I_v \leftrightarrow IV_v}{\sim} \left[\begin{array}{cccc|c} 6 & -3 & 7 & 8 & 9 \\ 2 & -1 & 3 & 4 & 5 \\ 4 & -2 & 5 & 6 & 7 \\ \lambda & -4 & 9 & 10 & 11 \end{array}\right] \overset{II_v \leftrightarrow I_v}{\sim}$$

$$\sim \left[\begin{array}{cccc|c} 2 & -1 & 3 & 4 & 5 \\ 6 & -3 & 7 & 8 & 9 \\ 4 & -2 & 5 & 6 & 7 \\ \lambda & -4 & 9 & 10 & 11 \end{array}\right] \overset{I_k \leftrightarrow IV_k}{\sim} \begin{array}{c} \begin{matrix} x_4 & x_2 & x_3 & x_1 & \end{matrix} \\ \left[\begin{array}{cccc|c} 4 & -1 & 3 & 2 & 5 \\ 8 & -3 & 7 & 6 & 9 \\ 6 & -2 & 5 & 4 & 7 \\ 10 & -4 & 9 & \lambda & 11 \end{array}\right] \end{array} \overset{I_k \leftrightarrow II_k}{\sim} \begin{array}{c} \begin{matrix} x_2 & x_4 & x_3 & x_1 & \end{matrix} \\ \left[\begin{array}{cccc|c} -1 & 4 & 3 & 2 & 5 \\ -3 & 8 & 7 & 6 & 9 \\ -2 & 6 & 5 & 4 & 7 \\ -4 & 10 & 9 & \lambda & 11 \end{array}\right] \end{array}$$

$$\underset{IV_v - I_v \cdot 4}{\overset{II_v - I_v \cdot 3,\ III_v - I_v \cdot 2}{\sim}} \left[\begin{array}{cccc|c} -1 & 4 & 3 & 2 & 5 \\ 0 & -4 & -2 & 0 & -6 \\ 0 & -2 & -1 & 0 & -3 \\ 0 & -6 & -3 & \lambda-8 & -9 \end{array}\right] \overset{II_k \leftrightarrow IV_k}{\sim} \begin{array}{c} \begin{matrix} x_2 & x_1 & x_3 & x_4 & \end{matrix} \\ \left[\begin{array}{cccc|c} -1 & 2 & 3 & 4 & 5 \\ 0 & 0 & -2 & -4 & -6 \\ 0 & 0 & -1 & -2 & -3 \\ 0 & \lambda-8 & -3 & -6 & -9 \end{array}\right] \end{array} \overset{III_v \leftrightarrow II_v}{\sim} \left[\begin{array}{cccc|c} -1 & 2 & 3 & 4 & 5 \\ 0 & 0 & -1 & -2 & -3 \\ 0 & 0 & -2 & -4 & -6 \\ 0 & \lambda-8 & -3 & -6 & -9 \end{array}\right]$$

$$\underset{IV_v - II_v \cdot 3}{\overset{III_v - II_v \cdot 2}{\sim}} \begin{array}{c} \begin{matrix} x_2 & x_1 & x_3 & x_4 & \end{matrix} \\ \left[\begin{array}{cccc|c} -1 & 2 & 3 & 4 & 5 \\ 0 & 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda-8 & 0 & 0 & 0 \end{array}\right] \end{array}$$

a) Za $\lambda=8$ imamo rang A = rang $\overline{A} = 2<4$ pa prema Kroneker-Kapelijevoj teoremi sistem ima ∞ mnogo rješenja.

2. promjenjive uzimamo proizvoljno npr. $x_4=t$ $x_1=s$

$$\begin{aligned} -x_3-2x_4 &= -3 \\ -x_2+2x_1+3x_3+4x_4 &= 5 \end{aligned}$$

$x_3=3-2t$

$-x_2+2s+3(3-2t)+4t=5$

$x_2=2s+9-6t+4t-5$

$x_2=2s-2t+4$

Za $\lambda=8$ rješenje sistema je $(s, 2s-2t+4, 3-2t, t)$ $t,s \in \mathbb{R}$

b) Za $\lambda \neq 8$ imamo rang A = rang $\overline{A} = 3<4$ pa prema Kroneker-Kapelijevom teoremu sistem ima ∞ mnogo rješenja.

1. (jednu) promjenjivu uzimamo proizvoljno npr. $x_4=t$

$$\begin{aligned} (\lambda-8)x_1 &= 0 \\ -x_3-2x_4 &= -3 \\ -x_2+2x_1+3x_3+4x_4 &= 5 \end{aligned}$$

$x_1=0$

$x_3=3-2t$

$-x_2+3(3-2t)+4t=5$

$x_2=9-6t+4t-5=-2t+4$

Za $\lambda \neq 8$ rješenje sistema je $(0, 4-2t, 3-2t, t)$.

Homogeni sistemi linearnih jednačina

(Zadaci su skinuti sa stranice: \pf.unze.ba\nabokov
Za uočene greške pisati na **infoarrt@gmail.com**)

Homogeni sistem linearnih jednačina je oblika $A\cdot x = 0$
gdje je

$$A=\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn}\end{bmatrix}_{m\times n}, \quad x=\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n\end{bmatrix}_{n\times 1} \quad \text{i} \quad 0=\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0\end{bmatrix}_{m\times 1}$$

Teorema: Homogeni sistem ima netrivijalna rješenja akko je $D=0$ ($\det A = 0$).

1. Riješiti homogeni sistem jednačina

$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \quad (1)\\ 3x_1 + x_2 - x_3 &= 0 \quad (2)\\ 2x_1 + x_2 &= 0 .\end{aligned}$$

Rj. $(1)+(2)$

$$\begin{aligned} 4x_1+2x_2&=0\\ 2x_1+x_2&=0 \quad /\cdot 2\end{aligned}$$

$$\begin{aligned} 4x_1+2x_2&=0\\ 4x_1+2x_2&=0\end{aligned}$$

$4x_1+2x_2=0 \quad /:2$

$2x_1+x_2=0$

sistem ima ∞ mnogo rješenja

$x_2=-2x_1$

$x_1=t$, $x_2=-2t$, $t\in\mathbb{R}$

$t-2t+x_3=0$

$x_3=t$

Sistem ima beskonačno mnogo rješenja oblika $(t,-2t,t)$

2. Nađi λ tako da sistem

$$\begin{aligned} 3x + y + \lambda z &= 0\\ 4x - 8y + \lambda z &= 0\\ 5x - 3y + 3z &= 0\end{aligned}$$

ima netrivijalna rješenja pa nađi rješenja.

Rj.

$$D=\begin{vmatrix} 3 & 1 & \lambda \\ 4 & -8 & \lambda \\ 5 & -3 & 3\end{vmatrix} \underset{III_v+I_v 3}{\overset{II_v+I_v 8}{=\!=}} \begin{vmatrix} 3 & 1 & \lambda \\ 28 & 0 & 9\lambda \\ 14 & 0 & 3\lambda+3\end{vmatrix} = -\begin{vmatrix} 28 & 9\lambda \\ 14 & 3\lambda+3\end{vmatrix} = (-14)\cdot 3\begin{vmatrix} 2 & 3\lambda \\ 1 & \lambda+1\end{vmatrix} = -42(-\lambda+2)$$

Za $\lambda=2$ ($D=0$) u sistemu postoje netrivijalna rješenja.

Sistem sad izgleda:

$$\begin{aligned} 3x+y+2z&=0 \quad /\cdot 3\\ 4x-8y+2z&=0 \quad /\cdot 3\\ 5x-3y+3z&=0 \quad /\cdot 2\end{aligned}$$

$$\begin{aligned} 9x+3y+6z&=0 \quad (1)\\ 12x-24y+6z&=0 \quad (2)\\ 10x-6y+6z&=0 \quad (3)\end{aligned}$$

$(3)-(1)$: $x-9y=0$

$(2)-(1)$: $3x-27y=0 \quad /:3$

$x-9y=0$

$x=9y$, $z=-14y$ postoji ∞ mnogo rješenja

$(9t, t, -14t)$, $t\in\mathbb{R}$ su rješenja sistema

3. Za koje vrijednosti λ sistem ima netrivijalna rješenja

$$\begin{aligned} \lambda x_1 + x_2 + x_3 + x_4 &= 0\\ x_1 + \lambda x_2 + x_3 + x_4 &= 0\\ x_1 + x_2 + \lambda x_3 + x_4 &= 0\\ x_1 + x_2 + x_3 + \lambda x_4 &= 0 .\end{aligned}$$

Rj. za $\lambda=1$ ili $\lambda=-3$